CONCERNING APPROACHABILITY OF SIMPLE CLOSED AND OPEN CURVES*

BY

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Schoenflies† was the first to formulate the converse of the fundamental theorem of C. Jordan‡ that a simple closed curve§ lying wholly within a plane decomposes the plane into an inside and an outside region. The statement of this converse theorem is as follows: *Suppose K is a closed, bounded set of points lying in a plane S and that $S - K = S_1 +. S_2$, where S_1 and S_2 are point-sets such that* (1) *every two points of S_i $(i = 1, 2)$ can be joined by an arc lying entirely in S_i* (2) *every arc joining a point of S_1 to a point of S_2 contains at least one point of K* (3) *if O is a point of K and P is a point not belonging to K, then P can be joined to O by an arc that has no point except O in common with K. Every point-set that satisfies these conditions is a simple closed curve.* Schoenflies used metrical hypotheses in his proof. Lennes gave a proof of this converse theorem using straight lines.‖ R. L. Moore pointed out that a proof similar in large part to that of Lennes can be carried through with the use of arcs and closed curves on the basis of his system of postulates Σ_3, thus furnishing a non-metrical proof of the converse theorem.¶

In all these proofs the condition numbered three, the condition of approachability (erreichbarkeit) plays a fundamental rôle. It is the purpose of the present paper to study the effect of substituting for the condition of approach-

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† Cf. A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Nachrichten der Göttinger Gesellschaft der Wissenschaften, 1902, p. 185.

‡ C. Jordan, *Cours d'Analyse*, 2d ed., Paris, 1893, p: 92.

§ If A and B are distinct points, then a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B, but containing no proper connected subset containing both A and B, Cf. N. J. Lennes *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), p. 308. A *simple closed curve* is a set of points composed of two arcs AXB and AYB having no point in common other than A and B. Hereafter in this paper "arc" and "closed curve" will be considered synonymous with "simple continuous arc" and "simple closed curve," respectively.

‖ Cf. N. J. Lennes, loc. cit., § 5.

¶ Cf. R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 59.

ability, the condition that the set is "*connected in kleinem*."* The results obtained are embodied in the following theorem:

THEOREM A. *Suppose K is a closed plane point-set, S is the set of all points of the plane, while $S - K = S_1 + S_2$, where S_1 and S_2 are two mutually exclusive domains*† *such that every point of K is a common boundary point of S_1 and S_2. Then a necessary and sufficient condition that K be either a simple closed curve or an open curve*‡ *is that K be connected in kleinem.*

That the condition stated in Theorem A is necessary is evident. I will proceed to show that it is sufficient. Suppose K is a connected in kleinem set satisfying the conditions stipulated in Theorem A. Then the following lemmas hold true:

LEMMA A. *Every arc joining a point of S_1 to a point of S_2 contains a point of K.*

Proof. Suppose it were possible to draw an arc from a point P_1 of S_1 to a point P_2 of S_2 that contains no point of K. Then let us divide the arc $P_1 P_2$ into two sets, M_1 and M_2, where M_1 is the set of all points of $P_1 P_2$ that belong to S_1, while M_2 is the set of all points of $P_1 P_2$ which belong to S_2. As $P_1 P_2$ is a connected point-set either M_1 contains a limit point of M_2 or M_2 contains a limit point of M_1.

Case I. A point F of M_1 is a limit point of M_2. As F is a point of the domain S_1, there exists a region containing F and lying entirely in S_1. As S_1 and S_2 are mutually exclusive domains, this region contains no point of M_2. Hence F cannot be a limit point of M_2.

Case II. A point G of M_2 is a limit point of M_1. This is impossible as in Case I.

Hence we are led to a contradiction if we suppose our lemma false.

LEMMA B. *The set K is connected.*

Proof. Suppose K were not connected. Then it could be divided into two mutually exclusive sets K_1 and K_2, neither of which contains a limit point of the other one. Let P_i $(i = 1, 2)$ denote a point of K_i. Put about P_i a circle R_i having P_i as center and such that R_i and its interior lie entirely

* Cf. Hans Hahn, *Ueber die allgemeinste ebene Punktmenge die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23 (1914), pp. 318–22. According to Hahn, a set of points C is said to be *connected in kleinem* if, whenever P is a point of C, ϵ a positive number and K a circle of radius $1/\epsilon$ with center at P, then there exists within K and with center at P, another circle $K_{\epsilon, P}$ such that if X is a point of C within $K_{\epsilon, P}$ then X and P lie together in some connected subset of C that lies entirely within K.

† A *domain* is a connected set of points M such that if P is a point of M, then there is a region that contains P and lies in M.

‡ An *open curve* is defined by R. L. Moore as a closed, connected, set of points M such that if P is a point of M, then $M - P$ is the sum of two mutually exclusive connected point-sets, neither of which contains a limit point of the other.

without R_{i+1}.* As K is connected in kleinem, there exists a circle $\bar{R}$, lying within R_i and with center at P_i such that if X_i is a point of K within $\bar{R}_i$, then X_i and P_i lie on some connected subset of K lying within R_i. It may easily be shown that X_i can be joined to P_i by a simple continuous arc of K lying entirely within R_i.† As every point of K is a common boundary point of S_1 and S_2, then there exists within $\bar{R}_i$ a point M_{ij} $(j = 1, 2)$ belonging to S_j. As S_j is a domain, then there exists a simple continuous arc $M_{1j} K_j M_{2j}$ lying entirely in S_j. Join M_{ij} to P_i by a simple continuous arc $M_{ij} L_{ij} P_i$ lying entirely within $\bar{R}_i$ and let G_{ij} denote the first point of K on the arc $M_{ij} L_{ij} P_i$ following M_{ij}. Then we may join G_{i1} to G_{i2} by an arc $G_{i1} F_i G_{i2}$ belonging to K and lying entirely within R_i. The point-set $G_{11} M_{11}$ (on $M_{11} L_{11} P_1$) $+ M_{11} K_1 M_{21} + M_{21} G_{21}$ (on $M_{21} L_{21} P_2$) contains as a subset a simple continuous arc $G_{11} H_1 G_{21}$ lying except for its endpoints entirely in S_1, while the set $G_{12} M_{12}$ (on $M_{12} L_{12} P_1$) $+ M_{12} K_2 M_{22} + M_{22} G_{22}$ (on $M_{22} L_{22} P_2$) contains as a subset a simple continuous arc $G_{12} H_2 G_{22}$ lying except for its endpoints entirely in S_2. We then have a closed curve $G_{11} F_1 G_{12} H_2 G_{22} - F_2 G_{21} H_1 G_{11}$ such that the arcs $G_{11} F_1 G_{12}$ and $G_{21} F_2 G_{22}$ lie entirely on K and within R_1 and R_2, respectively, while $\underline{G_{11} H_1 G_{21}}$‡ and $\underline{G_{12} H_2 G_{22}}$ belong to S_1 and S_2, respectively.

All points of $G_{11} F_1 G_{12}$ belong to K_1. For suppose a point H of $G_{11} F_1 G_{12}$ belonged to K_2. As H is joined to G_{11}, which in turn can be joined to P_1 by an arc of K lying entirely within R_1, it follows that H can be joined to P_1 by an arc HFP_1 of K lying entirely within R_1. Let $[\bar{H}_1]$ denote the set of all points of HFP_1 belonging to K_1 while $[\bar{H}_2]$ denotes the set of all points of HFP_1 belonging to K_2. Clearly neither of these sets contains a limit point of the other. Hence the arc HFP_1 is not a connected point-set. Hence the supposition that H belongs to K_2 has led to a contradiction. In like manner, all points of $G_{21} F_2 G_{22}$ belong to K_2.

The interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ must contain at least one point of K. For suppose it does not contain a point of K. Then the interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ is a subset of $S_1 + S_2$. Suppose it contains a point H of S_1. Then H can be joined to H_2 by an arc HXH_2 lying except for H_2 entirely within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$.§ Let $[W_1]$ denote the set of all points of HXH_2 belonging to S_1 while $[\dot{W}_2]$ denotes the set of all points of HXH_2 which are points of S_2. Clearly neither of these sets contains

* It is understood that subscripts are reduced modulo 2,

† Cf. R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917). While Professor Moore's theorem states that every two points of a continuous curve can be joined by a simple continuous arc lying entirely on the given continuous curve, it is clear that his methods suffice to prove the above stronger statement.

‡ If AXB is an arc, then the symbol $\underline{AXB}$ will denote $AXB - A - B$,

§ Cf. R. L. Moore, *Foundations of plane analysis situs*, loc. cit., Theorem 39, pp. 153–5.

a limit point of the other. Hence the arc HXH_2 is not a connected point-set. In like manner the supposition that there is within $G_{11} F_1 G_{12} H_2 G_{22} F_2$-$G_{21} H_1 G_{11}$, a point of S_2 leads to a contradiction. Hence $G_{11} F_1 G_{12} H_2 G_{22} F_2$-$G_{21} H_1 G_{11}$ must enclose a point of K.

Let $[V_2]$ denote the set of all points V_2 such that either (1) V_2 is a point of $G_{21} F_2 G_{22}$, or (2) V_2 is a point such that there exists a closed connected set $V_2 XF_2'$ belonging to K and lying within or on $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and such that F_2' is a point of $G_{21} F_2 G_{22}$. As K is connected in kleinem it may easily be proved that $[V_2]$ is a closed set. It is also true that all points of $[V_2]$ belong to K_2. Hence no point of $G_{11} F_1 G_{12}$ either belongs to or is a limit point of $[V_2]$. It may also be proved with the use of the in kleinem property that no point of $[V_2]$ is a limit point of a set of points of K lying within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and containing no point of V_2. There exists an arc $H_1 YH_2$ such that (1) $H_1 YH_2$ is a subset of the interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and (2) $\overline{H_1 YH_2}$ contains no points of $[V_2]$.* Let $[V_1]$ denote the set of all points of K within or on the closed curve, $H_1 YH_2 G_{22} F_2 G_{21} H_1$, not belonging to $[V_2]$. The set $[V_1]$ is closed. Put about each point of $[V_1]$ a circle lying entirely within $G_{11} F_1 G_{12} H_2 G_{22} F_2$-$G_{21} H_1 G_{11}$ and containing within it or on its boundary no point of $[V_2]$. By the Heine-Borel Property, there exists a finite number of circles of the above set, $C_1, C_2, \cdots, C_n$, covering $[V_1]$. With the use of Theorems 41, 42, 43, and 44 of Professor Moore's Foundations we may easily obtain from the set $C_1, C_2, \cdots, C_n$ and the closed curve $G_{11} F_1 G_{12} H_2 YH_1 G_{11}$, a new closed curve $G_{11} F_1 G_{12} H_2 ZH_1 G_{11}$, where the arc $H_1 G_{11} F_1 G_{12} H_2$ of the new closed curve $G_{11} F_1 G_{12} H_2 ZH_1 G_{11}$ is the arc $H_1 G_{11} F_1 G_{12} H_2$ of $G_{11} F_1 G_{12} H_2 YH_1 G_{11}$ and where $H_2 ZH_1$ is free from points of K and lies within $G_{11} F_1 G_{12} H_2 G_{22} F_2$-$G_{21} H_1 G_{11}$. But then we have a point of S_1 joined to a point of S_2 by an arc containing no point of K. Thus the supposition that K is not connected, leads to a contradiction.

LEMMA C. *If K contains one simple closed curve J, then all points of K belong to J.*

Proof. Suppose Lemma C is not true. Then K contains a closed curve J and at least one point P not on J. Two cases may arise:

Case I. P is within J. As every point of K is a common boundary point of S_1 and S_2, the interior of J contains a point P_1 of S_1 and a point P_2 of S_2. The exterior of J cannot contain a point $\overline{P}_1$ of S_1. For suppose it did. Then any arc from P_1 to $\overline{P}_1$ would contain a point of J and hence a point of K, contrary to the fact that S_1 is a domain. In like manner no point of S_2 can be in the exterior of J. Hence the exterior of J must be a subset of K, while

* Cf. my paper, *A definition of sense on plane closed curves in non-metrical analysis situs*, Annals of Mathematics, vol. XIX (1918), Theorem D, pp. 188–9.

S_1 and S_2 are subsets of the interior of J. But this is impossible because no point without J is a limit point of a set of points lying entirely within J thus making it impossible that every point of K be a common boundary point of S_1 and S_2. Hence the supposition that P is within J has led to a contradiction.

Case II. P is without J. Case II may be proved impossible by an argument similar to that used in Case I.

An immediate consequence of Lemma C is that if K is not a simple closed curve, then there is but one K-arc from a point A of K to a distinct point B of K.

LEMMA D. *The set K does not contain three arcs OP_1, OP_2, and OP_3, no two of which have a common point other than O.*

Proof. Suppose Lemma D were false. Then there would exist three arcs OP_1, OP_2, and OP_3, no two of which have a point in common other than O. Put about P_i $(i = 1, 2, 3)$ a circle C_i such that the point-set OP_{i+1}* $+ OP_{i+2}$ is a subset of the exterior of C_i and such that C_i has no point in common with $C_{i+1} + C_{i+2}$. As K is connected in kleinem, there exists within C_i and with center at P_i, another circle C_{P_i, C_i} such that if X_i is a point of K within C_{P_i, C_i}, then there is an arc from X_i to P_i every point of which is a point of K and which lies entirely within C_i.† As all points of K are limit points of both S_1 and S_2, C_{P_i, C_i} must contain at least one point $P_{i,1}$ of S_1. As S_1 is a domain, there is an arc $P_{11} P_{21}$ from P_{11} to P_{21} all points of which belong to S_1. Join $P_{i,1}$ to P_i by an arc $P_{i,1} P_i$ lying entirely within C_{P_i, C_i} and let X_i denote the first point of the arc $P_{i,1} P_i$ after $P_{i,1}$, which belongs to K. There exists an arc $X_i P_i$ from X_i to P_i belonging to K and lying entirely within C_i. Let P'_i denote the first point of the arc $X_i P_i$ which is on OP_i. The point-set $P'_1 X_1 + X_1 P_{11} + P_{11} P_{21} + P_{21} X_2 + X_2 P'_2$ contains as a subset an arc $P'_1 F_1 P'_2$ such that (1) $P'_1 F_1 P'_2$ has no point in common with $OP_1 + OP_2 + OP_3$, (2) all points of $P'_1 F_1 P'_2$ belong to either K or S_1, (3) at least one point, F_1, of S_1 is a point of $P'_1 F_1 P'_2$. By methods similar to those just employed, we may construct an arc $Q'_1 H_2 Q'_2$ from a point Q'_1 of OP_1 to a point Q'_2 of OP'_2 such that (1) Q'_i is on OP_i between P'_i and O, (2) all points of $Q'_1 H_2 Q'_2$ belong to either S_2 or K, (3) except for Q'_1 and Q'_2, $Q'_1 H_2 Q'_2$ has no point in common with $P'_1 FP'_2 + OP_1 + OP_2 + OP_3$, (4) at least one point H_2 of $Q'_1 H_2 Q'_2$ belongs to S_2. Two cases may arise:

Case I. $Q'_1 H_2 Q'_2$ is entirely within $OP'_1 F_1 P'_2 O$. Then the interior of $OP'_1 F_1 P'_2 O = Q'_1 H_2 Q'_2 +$ the interior of $OQ'_1 H_2 Q'_2 O +$ the interior of $P'_1 F_1 P'_2 Q'_2 H_2 Q'_1 P'_1$. The point-set $OP_3 + P_3$ is either entirely within or entirely without $OQ'_1 H_2 Q'_2 O$.

(*a*) Suppose $OP_3 + P_3$ is entirely within $OQ'_1 H_2 Q'_2 O$. Then $OQ'_1 H_2 Q'_2 O$

* It is understood throughout this argument that subscripts are reduced modulo 3.

† See an earlier footnote.

must enclose at least one point L of S_1. But then an arc from L to F_1 must contain at least one point of $OQ_1' H_2 Q_2' O$. Hence, as $OQ_1' H_2 Q_2' O$ is a subset of $K + S_2$, no such arc LF_1 can lie entirely in S_1, contrary to the fact that S_1 is a domain.

(*b*) Suppose $\underline{OP_3} + P_3$ is entirely without $OQ_1' H_2 Q_2' O$. It follows that $\underline{OP_3} + P_3$ is entirely without $OP_1' F_1 P_2' O$. Then the exterior of $OP_1' F_1 P_2' O$ contains at least one point M of S_2. Then any arc from M to H_2 must contain at least one point of $OP_1' F_1 P_2' O$ and hence at least one point not in S_2. But this is contrary to the fact that S_2 is a domain.

Thus in Case I we are led to a contradiction.

Case II. $\underline{Q_1' H_2 Q_2'}$ is without $OP_1' F_1 P_2' O$. We may show that Case II is impossible by methods similar to those used in Case I.

LEMMA *E. If O is a point of K and P is a point of S_i* ($i = 1, 2$) *then there exists at least one arc OP such that $\underline{OP} + P$ is a subset of S_i.*

Proof. Two conceivable cases may arise.

Case I. There exist points A_1 and A_2 of $K\,[A_1 \neq O \neq A_2]$ such that O is a point of the arc $A_1 O A_2$ belonging to K. By the same methods as were used in the preceding lemma we may construct an arc $A_1' F_1 A_2'$ such that (1) on $A_1 O A_2$ the order $A_1 A_1' O A_2' A_2$ holds, (2) $A_1' F_1 A_2'$ is a subset of $S_1 + K$, (3) at least one point F_1 of $A_1' F_1 A_2'$ is a point of S_1, (4) no point of $\overline{A_1' F A_2'}$ belongs to $A_1 O A_2$. The point O is not a limit point of $K - A_1' O A_2'$. For suppose it were. Then it would be a sequential limit point of a set of points $P_1, P_2, \cdots$, every one of which belongs to $K - A_1' O A_2'$. Put about O as center a circle M such that A_1' and A_2' are both without M. As K is connected in kleinem there exists another circle $\overline{M}$ lying within M and having its center at O such that if X is a point of K within $\overline{M}$, then X and O can be joined by an arc of K lying entirely within M. Let $\overline{P}$ denote that point of the set $P_1, P_2, \cdots$ of lowest subscript which lies within $\overline{M}$, while $\overline{P}O$ denotes an arc of K from $\overline{P}$ to O lying entirely within M. Let O' denote the first point of $\overline{P}O$ which is on $A_1' O A_2'$. Then the set K contains three arcs $A_1' O'$, $A_2' O'$, and PO', no two of which have a point in common other than O'. But this is contrary to Lemma D. Hence O cannot be a limit point of $K - A_1' O A_2'$. There exists a closed curve G enclosing O but enclosing no points of $A_1' F A_2' + [K - A_1' O A_2']$. Then there exist two closed curves J_1' and J_2' such that (1) every point of J_1' or J_2' belongs either to G or to $A_1' F_1 A_2' O A_1'$ (2) O is on J_1' and on J_2' (3) every point within J_1' is within $A_1' F_1 A_2' O A_1'$ while every point within J_2' is without $A_1' F_1 A_2' O A_1'$ (4) every point within either J_1' or J_2' is within G.* It is clear that either the interior of J_1' or the interior of J_2' is a subset of S_1 while the interior of the other of these two closed curves is a subset of S_2. Let J_1 denote that one whose interior is a subset of S_1 while

* Cf. R. L. Moore, *Foundations*, Theorem 43, pp. 156–7.

J_2 denotes the one whose interior is a subset of S_2. Let E denote a point within J_1, while P_1 is any other point of S_1. There exists an arc EO such that $EO - O$ is a subset of the interior of J_1.* As S_1 is a domain, there is an arc EP_1 lying entirely in S_1. The point-set $EO + EP_1$ contains as a subset an arc from P_1 to O lying except for O entirely in S_1. In like manner we may show that any point P_2 of S_2 can be joined to O by an arc lying except for O entirely in S_2.

Case II. There do not exist two distinct points A_1 and A_2 of K such that O is on an arc of K from A_1 to A_2. Let A denote a point of K different from O while ARO denotes an arc of K from A to O. By an argument similar to that employed in Case I we may show that if O were a limit point of $K - ARO$, then either there would exist three arcs AR', $R'O$, and $R'P$, no two of which have a point in common other than R' or there would exist a point A', $(A \neq A' \neq O)$ such that O is an arc of K from A' to A. But the first of these possibilities contradicts Lemma D while the second is contrary to the hypothesis of Case II. Hence O cannot be a limit point of $K - ARO$. Put about O a circle C that neither contains or encloses any point of $K - ARO$. Let $P_1, P_2, \cdots$ denote a set of points of S_1 approaching O as their sequential limit point. It is possible to pass at least one simple continuous arc† through $ARO + P_1 + P_2 + \cdots$. Let $P_1' ORA$ denote one such arc. If the interval OP_1' of the arc $P_1' ORA$ does not lie entirely with in C, let P' denote the first point which it has in common with C. Otherwise let P' denote P_1'. Let $\overline{P}$ denote that point of the set $P_1, P_2, \cdots$ of lowest subscript lying on OP'. It is clear that the sub-arc $O\overline{P}$ of $P_1' ORA$ lies, except for O, entirely in S_1. Let F_1 denote any other point of S_1. Join F_1 to $\overline{P}$ by an arc lying entirely in S_1. Then the point-set $O\overline{P} + \overline{P}F_1$ contains as a subset an arc from O to F_1, lying except for O entirely in S_1.

In like manner we may show that if F_2 is a point of S_2, F_2 can be joined to O by an arc lying except for O entirely in S_2.

LEMMA F. *A necessary and sufficient condition that K be bounded, is that either S_1 or S_2 be bounded.*

Proof. The condition is necessary. Let us suppose that K is bounded while neither S_1 nor S_2 is bounded. As K is bounded, there is a circle C such that all points of K are within C. As S_1 and S_2 are unbounded, there is a point P_1 of S_1 and a point P_2 of S_2 without C. Join P_1 and P_2 by an arc lying entirely without C. By Lemma A, this arc must contain a point of K. But all points of K are within C. Hence we are led to a contradiction if we suppose our condition is not necessary.

* Cf. R. L. Moore, *Foundations*, Theorem 39, pp. 153–5.

† Cf. R. L. Moore and J. R. Kline, *On the most general closed point-set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. XX (1919), pp. 218–23.

The condition is sufficient. For suppose S_1 is bounded while K is unbounded. Since S_1 is bounded, there exists a circle C enclosing S_1. Since K is unbounded, it contains a point P without C. The point P cannot be a limit point of S_1. But this is contrary to hypothesis.

Proof of Theorem A. Two cases may arise:

Case I. K is bounded. Then, by Schoenflies' Theorem and the preceding lemmas, it follows that K is a simple closed curve.

Case II. K is unbounded. It follows, by Lemma F that neither S_1 nor S_2 is bounded. Then K is an open curve. For a proof of this statement see my paper, "The converse of the theorem concerning the division of a plane by an open curve."*

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* Cf. these Transactions, vol. 18 (1917), pp. 177–184.